POINCARÉ SERIES AND MODULAR FUNCTIONS FOR $U(n,1)$

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1. Introduction

In the theory of automorphic forms, two classes of rank one reductive Lie groups $O(n,1)$ and $U(n,1)$ are the important objects. Automorphic forms on $O(n,1)$ have been intensively studied. In this paper we study the automorphic forms on $U(n,1)$. We construct infinitely many modular forms and non-holomorphic automorphic forms on $U(n,1)$ with respect to a discrete subgroup of infinite covolume. More precisely, we obtain the following theorem:

Main Theorem. *A function $f : \mathbb{H}_{\mathbf{C}}^{n+1} := \{Z = (z_1, \cdots, z_{n+1}) \in \mathbf{C}^{n+1} : Imz_{n+1} > \sum_{j=1}^{n} |z_j|^2\} \to \mathbf{C}$ is called a nonholomorphic automorphic form attached to the unitary group $U(n+1,1)$ if it satisfies the following three conditions:*

1. *f is an eigenfunction of the Laplace-Beltrami operator L of $U(n+1,1)$ on $\mathbb{H}_{\mathbf{C}}^{n+1}$;*
2. *f is invariant under the modualr group; i.e., $f(\gamma(Z)) = f(Z)$ for all $\gamma \in G(\mathbf{Z})$ and all $Z \in \mathbb{H}_{\mathbf{C}}^{n+1}$, where $G(\mathbf{Z})$ is the discrete subgroup of $G := TU(n+1,1)T^{-1}$ defined over $\mathbf{Z}$ and T is the Cayley transform from the unit ball $\mathbf{B}_{\mathbf{C}}^{n+1}$ in $\mathbf{C}^{n+1}$ to $\mathbb{H}_{\mathbf{C}}^{n+1}$.*
3. *f has at most polynomial growth at infinity; i.e., there are constants $C > 0$ and k such that $|f(Z)| \leq C\rho^k$, as $\rho \to \infty$ uniformly in t, for fixed β. Here $\rho(Z) = Imz_{n+1} - \sum_{j=1}^{n} |z_j|^2$ and $\beta(Z) = \sum_{j=1}^{n} |z_j|^2$.*

We denote by $\mathcal{N}(G(\mathbf{Z}), \lambda)$ the space of such nonholomorphic automorphic forms attached to $U(n+1,1)$. Then there exist a family of Poincaré series (hence, infinitely many elements) $r(Z, Z'; s) \in \mathcal{N}(G(\mathbf{Z}), s(s-n-1))$ for $Re(s) > n$, where

$$r(Z, Z'; s) := \sum_{\gamma \in G(\mathbf{Z})} g_s(x(Z, \gamma(Z')), y(Z)). \tag{1.1}$$

In the nondegenerate case,

$$g_s(x,y) = g_s(x,y;a,b) = x^{-a} y^{-b} F_3(a, b; a, b-n+1; 2s-n; -x^{-1}, -y^{-1}) \tag{1.2}$$

for $Re(a) > 1$, $Re(b) > n-1$ with $a+b = s$ and $F_3 = F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y)$ is a two variable hypergeometric function; in the degenerate case,

$$g_s(x,y) = \begin{cases} x^{-s} {}_2F_1(s, s; 2s-n; -x^{-1}), & \text{or} \\ (x+y)^{-s} {}_2F_1(s, s-n; 2s-n; -(x+y)^{-1}), \end{cases} \tag{1.3}$$

where ${}_2F_1 = {}_2F_1(\alpha, \beta; \gamma; z)$ *is a hypergeometric function. Here*

$$x(Z, Z') = \frac{(t(Z) - t(Z'))^2 + (\rho(Z) + \beta(Z) - \rho(Z') - \beta(Z'))^2}{4\rho(Z)(\rho(Z') + \beta(Z'))}, \quad y(Z) = \frac{\beta(Z)}{\rho(Z)} \tag{1.4}$$

are two invariants under the action of $G(\mathbf{Z})$*, where* $Z, Z' \in \mathbb{H}_{\mathbf{C}}^{n+1}$ *and* $t(Z) = Re(z_{n+1})$.

On the other hand, the Eisenstein series $E(Z; \lambda, \mu) \in \mathcal{N}(G(\mathbf{Z}), s(s-n-1))$*, where*

$$E(Z; \lambda, \mu) = \sum_{\gamma \in G(\mathbf{Z})_\infty \backslash G(\mathbf{Z})} \rho(\gamma(Z))^\lambda \beta(\gamma(Z))^\mu \tag{1.5}$$

with $(\lambda, \mu) = (s, 0)$ *or* $(s, 1-n)$.

A modular form on $U(n+1,1)$ *associated with* $G(\mathbf{Z})$ *is a function* $\phi : \mathbb{H}_{\mathbf{C}}^{n+1} \to \mathbf{C}$ *satisfying the following transform equations:*

(1) $\phi(\frac{z}{cz_{n+1}+d}, \frac{az_{n+1}+b}{cz_{n+1}+d}) = (cz_{n+1}+d)^k e^{2\pi imc(z_1^2+\cdots+z_n^2)/(cz_{n+1}+d)} \phi(z, z_{n+1})$*, for* $z = (z_1, \cdots, z_n)$.

(2) $\phi(wz, z_{n+1}) = \phi(z, z_{n+1})$ *for all* $w \in S_n$*, where* S_n *is the symmetric group of* n*-order.*

(3) $\phi(z, z_{n+1})$ *is a locally bounded function as* $Imz_{n+1} \to \infty$.

The set of modular forms on $U(n+1,1)$ *is denoted as* $M_{k,m}(G(\mathbf{Z}))$*. Then*

$$j_m(z, z_{n+1}) := \frac{1728 g_{2,m}(z, z_{n+1})^3}{\Delta_m(z, z_{n+1})} \in M_{0,0}(G(\mathbf{Z})), \tag{1.6}$$

where $g_{2,m_1}(z, z_{n+1}) := \frac{4}{3}\pi^4 E_{4,m_1}(z, z_{n+1})$, $g_{3,m_2}(z, z_{n+1}) := \frac{8}{27}\pi^6 E_{6,m_2}(z, z_{n+1})$*, and* $\Delta_m(z, z_{n+1}) := g_{2,m}(z, z_{n+1})^3 - 27 g_{3,\frac{3}{2}m}(z, z_{n+1})^2$*. Moreover,* $\{j_m\}_{m \in \mathbf{N}}$ *are a family of modular functions on the modular variety* $\mathcal{M}_{n+1} := G(\mathbf{Z}) \backslash \mathbb{H}_{\mathbf{C}}^{n+1}$.

2. The Laplace-Beltrami operator and the discrete subgroup of $U(n+1,1)$ on $\mathbb{H}_{\mathbf{C}}^{n+1}$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n+1,1)$, where $A = (a_{ij})_{(n+1)\times(n+1)}$, $B = (b_{ij})_{(n+1)\times 1}$, $C = (c_{ij})_{1\times(n+1)}$, $D = (d)_{1\times 1}$, and $W = (w_1, \cdots, w_{n+1}) \in \mathbf{B}_{\mathbf{C}}^{n+1} = \{W \in \mathbf{C}^{n+1} | |W|^2 = |w_1|^2 + \cdots + |w_{n+1}|^2 < 1\}$, the action of g on W is defined as $g \circ W = [(AW^t + B)(CW^t + D)^{-1}]^t$. According to [5], the corresponding Laplace-Beltrami operator on $U(n+1,1)$ is

$$L_{\mathbf{B}^{n+1}} = \mathrm{tr}[(I - WW^*)\overline{\partial_W} \cdot (I - W^*W) \cdot \partial_W'], \tag{2.1}$$

where ∂_W is the differential operator $\partial_W = \left(\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial w_{n+1}}\right)$, and the dots here indicate that the factor $(I - W^*W)$ is not differentiated.

Theorem 2.1. *The Laplace-Beltrami operator of* $U(n+1,1)$ *on* $\mathbf{B}_{\mathbf{C}}^{n+1}$ *is*

$$\Delta = (1 - \sum_{j=1}^{n+1} |w_j|^2) \left[\sum_{j=1}^{n+1} \frac{\partial^2}{\partial w_j \partial \overline{w_j}} - \left(\sum_{j=1}^{n+1} w_j \frac{\partial}{\partial w_j} \right) \left(\sum_{j=1}^{n+1} \overline{w_j} \frac{\partial}{\partial \overline{w_j}} \right) \right]. \tag{2.2}$$

Let $\mathbb{H}_{\mathbf{C}}^{n+1}$ be the Siegel domain of the second kind, i.e., the complex hyperbolic space, $\mathbb{H}_{\mathbf{C}}^{n+1} = \{(z_1, \cdots, z_{n+1}) \in \mathbf{C}^{n+1} | \mathrm{Im} z_{n+1} > \sum_{j=1}^{n} |z_j|^2\}$. It is well-known that $\mathbb{H}_{\mathbf{C}}^{n+1}$ is holomorphically equivalent to $\mathbf{B}_{\mathbf{C}}^{n+1}$. The Cayley transform T from $\mathbf{B}_{\mathbf{C}}^{n+1}$ onto $\mathbb{H}_{\mathbf{C}}^{n+1}$ is given by $z_1 = \frac{iw_1}{1-w_{n+1}}, \cdots, z_n = \frac{iw_n}{1-w_{n+1}}, z_{n+1} = i\frac{1+w_{n+1}}{1-w_{n+1}}$.

Theorem 2.2. *The Laplace-Beltrami operator of $U(n+1,1)$ on the complex hyperbolic space $\mathbb{H}_{\mathbf{C}}^{n+1}$ is*

$$L = (Imz_{n+1} - \sum_{j=1}^{n} |z_j|^2) \left[\sum_{j=1}^{n+1} 2i \left(\overline{z_j} \frac{\partial^2}{\partial z_{n+1} \partial \overline{z_j}} - z_j \frac{\partial^2}{\partial \overline{z_{n+1}} \partial z_j} \right) + \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} \right]. \tag{2.3}$$

Let us consider $\mathbf{B}_{\mathbf{C}}^{n+1}$ as a symmetric space $U(n+1,1)/U(n+1) \times U(1)$. Set $G = TU(n+1,1)T^{-1}$. Let K be a maximal compact subgroup of G and Γ be a discrete subgroup of G. $\mathbb{H}_{\mathbf{C}}^{n+1} = G/K$ is invariant under the action of G.

Let H be a subgroup of $U(n+1,1)$, if $THT^{-1} = \Gamma$ is a discrete subgroup of G, then

$$\begin{pmatrix} iI_n & & \\ & i & i \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \begin{pmatrix} iI_n & & \\ & i & i \\ & -1 & 1 \end{pmatrix},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H$ and $\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \in \Gamma$. We have $\begin{pmatrix} iA & iB \\ SC & SD \end{pmatrix} = \begin{pmatrix} i\widetilde{A} & \widetilde{B}S \\ i\widetilde{C} & \widetilde{D}S \end{pmatrix}$, where $S = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$. Thus, $A = \widetilde{A}$, $B = -i\widetilde{B}S$, $C = iS^{-1}\widetilde{C}$, $D = S^{-1}\widetilde{D}S$. By $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n+1,1)$, we have

$$\widetilde{A}\widetilde{A}^* + \widetilde{B}J\widetilde{B}^* = I_n, \quad \widetilde{C}\widetilde{A}^* + \widetilde{D}J\widetilde{B}^* = 0, \quad \widetilde{C}\widetilde{C}^* + \widetilde{D}J\widetilde{D}^* = J,$$

where $J = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$. If Γ is defined over $\mathbf{Z}$, then $\widetilde{B} = \widetilde{C} = 0$, $\widetilde{A} \in O(n, \mathbf{Z})$, $\widetilde{D} \in SL(2, \mathbf{Z})$. In fact, $\widetilde{A}$ is a permutation matrix with elements 0 and ± 1. For simplicity, we only consider the permutation matrix with elements 0 and 1. We denote σ as the above permutation matrix, it can be identified with the element of the symmetric group of n-order S_n. Therefore,

$$\Gamma = \left\{ \gamma = \begin{pmatrix} \sigma & & \\ & a & b \\ & c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \right\} \tag{2.4}$$

where $\sigma \in S_n$ is a permutation on the set $\{z_1, \cdots, z_n\}$ as $\sigma(z_j) = z_{\sigma(j)}$. Thus

$$\gamma(z_1, \cdots, z_{n+1}) = \left(\frac{\sigma(z_1)}{cz_{n+1} + d}, \cdots, \frac{\sigma(z_n)}{cz_{n+1} + d}, \frac{az_{n+1} + b}{cz_{n+1} + d} \right). \tag{2.5}$$

We denote this group as $G(\mathbf{Z})$. Its parabolic subgroup is

$$G(\mathbf{Z})_\infty = \left\{ \begin{pmatrix} \sigma & & \\ & \pm 1 & n \\ & 0 & \pm 1 \end{pmatrix} : n \in \mathbf{N}, \sigma \in S_n \right\}.$$

In coordinates $(t, \rho, z_1, \cdots, z_n)$ where $t = \mathrm{Re} z_{n+1}$, $\rho = \mathrm{Im} z_{n+1} - \sum_{j=1}^n |z_j|^2$, we define $\rho(Z) = \mathrm{Im} z_{n+1} - \sum_{j=1}^n |z_j|^2$ and $\beta(Z) = \sum_{j=1}^n |z_j|^2$, then $\rho \circ \gamma(Z) = \frac{\rho(Z)}{|cz_{n+1}+d|^2}$ and $\beta \circ \gamma(Z) = \frac{\beta(Z)}{|cz_{n+1}+d|^2}$.

Denote Θ the inverse conjugate transpose, it is an automorphism of the Lie groups called the Cartan involution. Let $U(n+1,1) = K_U A_U N_U = \overline{N_U} A_U K_U$ be the Iwasawa decomposition, where

$$N_U = \left\{ n(z,t) = \begin{pmatrix} I_n & iz^t & -iz^t \\ i\overline{z} & 1 - \frac{|z|^2 - it}{2} & \frac{|z|^2 - it}{2} \\ i\overline{z} & -\frac{|z|^2 - it}{2} & 1 + \frac{|z|^2 - it}{2} \end{pmatrix} : z \in \mathbf{C}^n, t \in \mathbf{R} \right\},$$

$$\overline{N_U} = \Theta N_U,$$

$$A_U = \left\{ a(\zeta) = \begin{pmatrix} I_n & 0 & 0 \\ 0 & \mathrm{ch}\zeta & \mathrm{sh}\zeta \\ 0 & \mathrm{sh}\zeta & \mathrm{ch}\zeta \end{pmatrix} : \zeta \in \mathbf{R} \right\},$$

$$K_U = U(n+1) \times U(1).$$

Let $P_U = \overline{N_U} A_U$ be the semidirect product of $\overline{N_U}$ and A_U, where the action of A_U on $\overline{N_U}$ is given by

$$a(\zeta) : \overline{n}(z,t) \mapsto a(\zeta)^{-1} \overline{n}(z,t) a(\zeta) = \overline{n}(e^\zeta z, e^{2\zeta} t).$$

In coordinates $z = (z_1, \cdots, z_n)$, $t = \mathrm{Re} z_{n+1}$ and $\rho = \mathrm{Im} z_{n+1} - |z|^2$, $\mathbb{H}_{\mathbf{C}}^{n+1} = \{(z,t,\rho) : z \in \mathbf{C}^n, t \in \mathbf{R}, \rho > 0\}$, $\partial \mathbb{H}_{\mathbf{C}}^{n+1} = \{(z,t) = (z,t,0) : z \in \mathbf{C}^n, t \in \mathbf{R}\}$. We identify $\overline{N_U}$ and P with $\partial \mathbb{H}_{\mathbf{C}}^{n+1}$ and $\mathbb{H}_{\mathbf{C}}^{n+1}$ under the map that $\overline{n}(z,t)$ and $\overline{n}(z,t)a(\zeta)$ are identified with (z,t) and (z,t,ρ), respectively. Here $\rho = e^{2\zeta}$.

The multiplication of $\overline{N_U}$ (or $\partial \mathbb{H}_{\mathbf{C}}^{n+1}$) is given by

$$(z,t)(z',t') = (z + z', t + t' + 2\mathrm{Im} z\overline{z'}),$$

where $z\overline{z'} = \sum_{j=1}^n z_j \overline{z'_j}$. So $\overline{N_U}$ is the Heisenberg group $\mathbf{H}^n$. The delation of $\mathbf{H}^n$ is given by $\rho(z,t) = (\sqrt{\rho} z, \rho t)$, $\rho > 0$, which is consistent with the delation of $\mathbb{H}_{\mathbf{C}}^{n+1}$ given by $\rho(z, z_{n+1}) = (\sqrt{\rho} z, \rho z_{n+1})$. The multiplication is defined as

$$(z,t,\rho)(z',t',\rho') = (z + \sqrt{\rho} z', t + \rho t' + 2\sqrt{\rho} \mathrm{Im} z\overline{z'}, \rho\rho').$$

P_U is a locally compact nonunimodular group with the left Haar measure

$$d\sigma(z,t,\rho) = \rho^{-(n+2)} dm(z) dt d\rho,$$

where $dm(z)$ denotes the Lebesque measure of $\mathbf{C}^n$.

From the theory of integrals on quotients G/H where H is a closed subgroup of the Lie group G, we will need an understanding of the formula:

$$\int_G f(g)dg = \int_{G/H}\int_H f(gh)dhd\overline{g}.$$

Here dg and dh are Haar measures on G, H, respectively. And this formula defines the G-invariant measure $d\overline{g}$ on the quotient space G/H. Such an integral is determined up to a positive constant. Formula holds provided that both G and H are unimodular. So $d\sigma$ is G-invariant measure. We have

$$\begin{aligned}
&\mathrm{Vol}(G(\mathbf{Z})\backslash\mathbb{H}_{\mathbf{C}}^{n+1}) = \int_{G(\mathbf{Z})\backslash\mathbb{H}_{\mathbf{C}}^{n+1}} \rho^{-(n+2)}dtd\rho dm(z)\\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_{n+1}\int_{\sqrt{1-x_{n+1}^2}}^{\infty} dy_{n+1}\int_{\sum_{j=1}^n(x_j^2+y_j^2)<y_{n+1}} \frac{dx_1dy_1\cdots dx_ndy_n}{[y_{n+1}-\sum_{j=1}^n(x_j^2+y_j^2)]^{n+2}}\\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_{n+1}\int_{\sqrt{1-x_{n+1}^2}}^{\infty} dy_{n+1}\int_{S^{2n-1}} d\omega\int_0^{\sqrt{y_{n+1}}}\frac{r^{2n-1}dr}{(y_{n+1}-r^2)^{n+2}},
\end{aligned}$$

where $\int_0^{\sqrt{y_{n+1}}}\frac{r^{2n-1}dr}{(y_{n+1}-r^2)^{n+2}} = \frac{1}{2}\int_0^{y_{n+1}}\frac{(y_{n+1}-t)^{n-1}}{t^{n+2}}dt = \infty$. Therefore, the covolume of $G(\mathbf{Z})$ is infinite.

3. The eigenfunctions of L

In this section, we will solve the eigenfunctions of L.

By the transform $u_j = x_j, v_j = y_j (1 \leq j \leq n)$, $t = x_{n+1}, \rho = y_{n+1} - \sum_{j=1}^n(x_j^2 + y_j^2)$ and Theorem 2.2, we have the following theorem:

Theorem 3.1. *In coordinates (x_j, y_j, t, ρ), $x_j = Rez_j$, $y_j = Imz_j$, $t = Rez_{n+1}$, $\rho = Imz_{n+1} - \sum_{j=1}^n|z_j|^2$, $x_j \neq 0$ and $y_j \neq 0$,*

$$\begin{aligned}
L =&\rho[\frac{1}{4}\sum_{j=1}^n(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}) + (\rho+\sum_{j=1}^n(x_j^2+y_j^2))\frac{\partial^2}{\partial t^2}+\rho\frac{\partial^2}{\partial\rho^2}-n\frac{\partial}{\partial\rho}+\\
&\sum_{j=1}^n(y_j\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial y_j})\frac{\partial}{\partial t}].
\end{aligned}\tag{3.1}$$

In particular, if $f = \rho^s$, then $Lf = (\rho^2\frac{\partial^2}{\partial\rho^2} - n\rho\frac{\partial}{\partial\rho})\rho^s = s(s-n-1)\rho^s$.

The definition of cusp forms on $U(n+1,1)$ requires that $\int_0^1 fdt = 0$. The corresponding Fourier expansion: $f = \sum_a c(a)Z(a, x_1, y_1, \cdots, x_n, y_n, \rho)e^{2\pi iat}$. By transform $x_j = \sqrt{\beta_j}\cos\theta_j$, $y_j = \sqrt{\beta_j}\sin\theta_j (1 \leq j \leq n)$, we have the following theorem:

Theorem 3.2. *In coordinates $(\beta_j, \theta_j, t, \rho)$,*

$$L = \rho[\sum_{j=1}^n\beta_j\frac{\partial^2}{\partial\beta_j^2}+\rho\frac{\partial^2}{\partial\rho^2}+(\rho+\sum_{j=1}^n\beta_j)\frac{\partial^2}{\partial t^2}+\sum_{j=1}^n\frac{1}{4\beta_j}\frac{\partial^2}{\partial\theta_j^2}-\sum_{j=1}^n\frac{\partial^2}{\partial\theta_j\partial t}+\sum_{j=1}^n\frac{\partial}{\partial\beta_j}-n\frac{\partial}{\partial\rho}].\tag{3.2}$$

Set

$$f_{a,b} = u(a, b_1, \cdots, b_n, \rho) v(a, b_1, \cdots, b_n, \beta_1, \cdots, \beta_n) e^{2\pi i a t} e^{2i(\sum_{j=1}^n b_j \theta_j)},$$

by $Lf_{a,b} = \lambda f_{a,b}$, we have

$$\sum_{j=1}^n \beta_j \frac{1}{v} \frac{\partial^2 v}{\partial \beta_j^2} + \rho \frac{u''}{u}(\rho) - 4\pi^2 a^2 (\rho + \sum_{j=1}^n \beta_j) + \sum_{j=1}^n [-\frac{b_j^2}{\beta_j} + 4\pi a b_j + \frac{1}{v} \frac{\partial v}{\partial \beta_j}] - n \frac{u'}{u}(\rho) = \frac{\lambda}{\rho}.$$

Therefore,

$$\begin{aligned} &\rho \frac{u''}{u}(\rho) - 4\pi^2 a^2 \rho - n \frac{u'}{u}(\rho) - \frac{\lambda}{\rho} + 4\pi a \sum_{j=1}^n b_j \\ = &- \sum_{j=1}^n \beta_j \frac{1}{v} \frac{\partial^2 v}{\partial \beta_j^2} + 4\pi^2 a^2 \sum_{j=1}^n \beta_j + \sum_{j=1}^n \frac{b_j^2}{\beta_j} - \sum_{j=1}^n \frac{1}{v} \frac{\partial v}{\partial \beta_j} = k = \text{const.} \end{aligned}$$

(1) $\rho^2 u''(\rho) - n\rho u'(\rho) - [\lambda + (k - 4\pi a \sum_{j=1}^n b_j)\rho + 4\pi^2 a^2 \rho^2] u(\rho) = 0.$

Set $k = 4\pi a \sum_{j=1}^n b_j$, then we have $\rho^2 u''(\rho) - n\rho u'(\rho) - (\lambda + 4\pi^2 a^2 \rho^2) u(\rho) = 0$. Let $u(\rho) = \rho^{\frac{n+1}{2}} w(\rho)$, then $\rho^2 w''(\rho) + \rho w'(\rho) - [\lambda + (\frac{n+1}{2})^2 + 4\pi^2 a^2 \rho^2] w(\rho) = 0$. A solution is $w(\rho) = K_{s - \frac{n+1}{2}}(2\pi |a| \rho)$, $\lambda = s(s - n - 1)$. Here K-Bessel function $K_s(z)$ is defined as $K_s(z) = \frac{1}{2} \int_0^\infty \exp[-\frac{z}{2}(t + \frac{1}{t})] t^{s-1} dt$, for $\mathrm{Re}(z) > 0$, $\mathrm{Re}(s) > 0$.

(2) $\sum_{j=1}^n \beta_j \frac{\partial^2 v}{\partial \beta_j^2} + \sum_{j=1}^n \frac{\partial v}{\partial \beta_j} + 4\pi a \sum_{j=1}^n b_j v - \sum_{j=1}^n \frac{b_j^2}{\beta_j} v - 4\pi^2 a^2 \sum_{j=1}^n \beta_j v = 0.$

For simplicity, we consider the case that $b_j = 0$, $1 \leq j \leq n$.

(i)$v = v_1(\beta_1) \cdots v_n(\beta_n)$, then $\sum_{j=1}^n (\beta_j \frac{v_j''}{v_j}(\beta_j) + \frac{v_j'}{v_j}(\beta_j) - 4\pi^2 a^2 \beta_j) = 0$. A particular case is $\beta_j v_j''(\beta_j) + v_j'(\beta_j) - 4\pi^2 a^2 \beta_j v_j(\beta_j) = 0, 1 \leq j \leq n$. A solution is $v_j = K_0(2\pi |a| \beta_j)$.

(ii) $v = v(\beta)$ with $\beta = \beta_1 + \cdots + \beta_n$, then $\beta v''(\beta) + n v'(\beta) - 4\pi^2 a^2 \beta v(\beta) = 0$. Set $v(\beta) = \beta^{\frac{1-n}{2}} w(\beta)$, then $\beta^2 w''(\beta) + \beta w'(\beta) - [(\frac{n-1}{2})^2 + 4\pi^2 a^2 \beta^2] w(\beta) = 0$. A solution is $w(\beta) = K_{\frac{n-1}{2}}(2\pi |a| \beta)$.

We have the following theorem:

Theorem 3.3. *Two solutions (we call them the normal solutions) of $Lf = \lambda f$ are*

$$\begin{aligned} f &= \sum_{m=0}^\infty a_m \rho^{\frac{n+1}{2}} K_{s - \frac{n+1}{2}}(2\pi |m| \rho) \prod_{j=1}^n K_0(2\pi |m| \beta_j) e^{2\pi i m t}, \quad \textit{and} \\ g &= \sum_{m=0}^\infty b_m \rho^{\frac{n+1}{2}} K_{s - \frac{n+1}{2}}(2\pi |m| \rho) \beta^{\frac{1-n}{2}} K_{\frac{n-1}{2}}(2\pi |m| \beta) e^{2\pi i m t}, \end{aligned} \tag{3.3}$$

where $\lambda = s(s - n - 1)$.

Set $f = \phi(\beta, t, \rho)$ with $\beta = \sum_{i=1}^n \beta_i$, then

$$Lf = \rho[\beta \frac{\partial^2}{\partial \beta^2} + \rho \frac{\partial^2}{\partial \rho^2} + (\rho + \beta) \frac{\partial^2}{\partial t^2} + n \frac{\partial}{\partial \beta} - n \frac{\partial}{\partial \rho}] \phi.$$

Set $\phi_s = \rho^s\beta^\nu$, then $L\phi_s = s(s-n-1)\phi_s + t(t+n-1)\rho^{s+1}\beta^{\nu-1}$. When $\nu = 0$ or $\nu = 1-n$, $L\phi_s = s(s-n-1)\phi_s$. Let $\phi_{s,0} := \rho(Z)^s$, $\phi_{s,1-n} := \rho(Z)^s\beta(Z)^{1-n}$. The Eisenstein series

$$E(Z;s,0) := \sum_{\gamma\in G(\mathbf{Z})_\infty\backslash G(\mathbf{Z})} \phi_{s,0}\circ\gamma, \quad E(Z;s,1-n) := \sum_{\gamma\in G(\mathbf{Z})_\infty\backslash G(\mathbf{Z})} \phi_{s,1-n}\circ\gamma \tag{3.4}$$

satisfy the following:

Theorem 3.4.

$$LE(Z;s,0) = s(s-n-1)E(Z;s,0), \quad LE(Z;s,1-n) = s(s-n-1)E(Z;s,1-n). \tag{3.5}$$

Let $f = f(\tau,\omega)$ with $\omega = \sqrt{\rho}$ and $\tau = (x_1,\cdots,x_n,y_1,\cdots,y_n)$, then

$$Lf = \frac{1}{4}[\omega^2(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}+\frac{\partial^2}{\partial y_1^2}+\cdots+\frac{\partial^2}{\partial y_n^2}+\frac{\partial^2}{\partial \omega^2}) - (2n+1)\omega\frac{\partial}{\partial\omega}]f. \tag{3.6}$$

Denote $e(x) = \exp(2\pi ix)$ and set $f_a = u(\omega)e(a\cdot\tau)$ with $a = (a_1,\cdots,a_{2n}) \in \mathbf{Z}^{2n}$ and $|a| := (\sum_{j=1}^{2n} a_j^2)^{\frac{1}{2}}$. By $Lf_a = \lambda f_a$, we have $\omega^2 u''(\omega) - (2n+1)\omega u'(\omega) - (4\lambda + 4\pi^2|a|^2\omega^2)u(\omega) = 0$. Let $u(\omega) = \omega^{n+1}w(\omega)$, then $\omega^2 w''(\omega) + \omega w'(\omega) - [4\lambda + (n+1)^2 + 4\pi^2|a|^2\omega^2]w(\omega) = 0$. A solution is $w(\omega) = \omega^{n+1}K_{2s-(n+1)}(2\pi|a|\omega)$ with $\lambda = s(s-n-1)$.

Theorem 3.5. *The other solution of $Lf = \lambda f$ (we call it the singular solution) is*

$$f = \sum_{a\in\mathbf{Z}^{2n}} A(a)\omega^{n+1}K_{2s-(n+1)}(2\pi|a|\omega)e(a\cdot\tau), \tag{3.7}$$

where $\lambda = s(s-n-1)$.

It is known that the Cygan metric ρ_c attached to $U(n+1,1)$ is given by (see [1]) $||(z,\rho,t)||_c = |||z||^2+\rho-it|^{\frac{1}{2}}$, for $(z,\rho,t)\in\mathbf{C}^n\times\mathbf{R}\times(0,\infty)$. We define the pseudo-distance d^* as follows

$$d^*(Z,Z') := \log||(z,t,\rho)^{-1}(z',t',\rho')||_c = \log\frac{1}{\sqrt{\rho}}||z-z'|^2+\rho'+i(t-t'+2\mathrm{Im}(z\overline{z'}))|^{\frac{1}{2}}.$$

Theorem 3.6. *The distance function is noneuclidean harmonic, i.e.,*

$$Ld^*((z',t',\rho'),(z,t,\rho)) = 0. \tag{3.8}$$

4. The integral transform of Eisenstein series on $U(n+1,1)$

In this section we will prove the following theorem:

THEOREM 4.1. *(The integral transform property of Eisenstein series on $U(n+1,1)$): The Eisenstein series have the following Fourier expansions:*

$$E(Z;s,0)=\sum_m a_m(\rho,\beta)e^{2\pi imt}, \quad E(Z;s,1-n)=\sum_m b_m(\rho,\beta)e^{2\pi imt}.$$

(1)Let

$$a_m(\rho):=\int_{(\mathbf{R}_+)^n}[a_m(\rho,\beta)-\delta_{0,m}\rho^s]\prod_{j=1}^n e^{-2\pi|m|\beta_j}G(2\pi|m|\beta_j)d\beta_1\cdots d\beta_n,$$

where $G(z)=\sum_{k=0}^\infty \frac{1}{2^k k!}C_{2k}^k z^k$ and $\delta_{i,j}$ is the Kronecker symbol. Then

$$a_m(\rho)=\begin{cases} 2^{1-n}\pi^{s-\frac{n}{2}}\dfrac{\Gamma(s-\frac{n}{2})}{\Gamma(s)^2}\varphi_m(s)|m|^{s-\frac{n+1}{2}}\rho^{\frac{n+1}{2}}K_{s-\frac{n+1}{2}}(2\pi|m|\rho),(m\neq 0),\\ 2^{-n}\sqrt{\pi}\dfrac{\Gamma(s-\frac{n}{2})\Gamma(s-\frac{n+1}{2})}{\Gamma(s)^2}\varphi_0(s)\rho^{n+1-s},(m=0),\end{cases}\tag{4.1}$$

with $\varphi_m(s)=\sum_{c>0}\frac{1}{|c|^{2s}}(\sum_{(d,c)=1,d\bmod c}e(\frac{md}{c}))$, where $e(x)$ denotes $\exp(2\pi ix)$.
(2)For $m\neq 0$, let

$$b_m(\rho)=\int_0^\infty b_m(\rho,\beta)e^{-2\pi|m|\beta}H(2\pi|m|\beta)d\beta,$$

where $H(z)=\sum_{k=n-1}^\infty \frac{2^k(1-\frac{n}{2},k)}{k!(k-n+1)!}z^k$, then

$$\begin{aligned} b_m(\rho)=&2^{n-1}\pi^{s-\frac{n}{2}}\frac{\Gamma(s-\frac{n}{2})}{\Gamma(s-n+1)^2}\varphi_m(s-n+1)|m|^{s-\frac{n+1}{2}}\rho^{\frac{n+1}{2}}K_{s-\frac{n+1}{2}}(2\pi|m|\rho)\\ &-R(|m|,\rho)\end{aligned}\tag{4.2}$$

with

$$\begin{aligned} R(|m|,\rho)=&2^{n-2}\pi^{s-\frac{n}{2}}\Gamma(s-n+1)^{-1}\varphi_m(s-n+1)|m|^{s-1-\frac{n}{2}}\rho^{\frac{n}{2}}\\ &\times\sum_{k=0}^{n-2}\frac{(1-\frac{n}{2},k)}{k!}(4\pi|m|\rho)^{\frac{k}{2}}W_{\frac{n}{2}-1-\frac{k}{2},s-\frac{n+1}{2}-\frac{k}{2}}(4\pi|m|\rho).\end{aligned}\tag{4.3}$$

Here $(\alpha,k):=\alpha(\alpha+1)\cdots(\alpha+k-1)$ and $W_{\kappa,\mu}(x)$ is the Whittaker function.

Proof. We have

$$\begin{aligned} a_m(\rho,\beta)=&\int_0^1\frac{1}{2}\sum_{c,d\in\mathbf{Z},(c,d)=1}\frac{\rho^s}{[(ct+d)^2+c^2(\rho+\beta)^2]^s}e^{-2\pi imt}dt\\ =&\delta_{0,m}\rho^s+\sum_{c>0}\frac{1}{|c|^{2s}}\int_{-\infty}^\infty\sum_{(d,c)=1,d\bmod c}\frac{\rho^s}{[(t+\frac{d}{c})^2+(\rho+\beta)^2]^s}e^{-2\pi imt}dt\\ =&\delta_{0,m}\rho^s+\sum_{c>0}\frac{1}{|c|^{2s}}\sum_{(d,c)=1,d\bmod c}e(\frac{md}{c})\int_{-\infty}^\infty\frac{\rho^s e^{-2\pi imt}}{[t^2+(\rho+\beta)^2]^s}dt.\end{aligned}$$

Set $\varphi_m(s)=\sum_{c>0}\frac{1}{|c|^{2s}}(\sum_{(d,c)=1,d\text{mod}c}e(\frac{md}{c}))$, by (see [7], p. 15)

$$\int_{-\infty}^{\infty}\frac{e(-ut)}{(1+t^2)^s}dt=2\pi^s|u|^{s-\frac{1}{2}}\Gamma(s)^{-1}K_{s-\frac{1}{2}}(2\pi|u|),\quad (u\neq 0,u\in\mathbf{R}),\tag{4.4}$$

we have for $m\neq 0$,

$$\int_{-\infty}^{\infty}\frac{\rho^s e^{-2\pi imt}}{[t^2+(\rho+\beta)^2]^s}dt=\frac{2\pi^s}{\Gamma(s)}|m|^{s-\frac{1}{2}}\rho^s(\rho+\beta)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|m|(\rho+\beta)).$$

So,

$$\begin{aligned}a_m(\rho)=&\varphi_m(s)\frac{2\pi^s}{\Gamma(s)}|m|^{s-\frac{1}{2}}\rho^s\int_0^{\infty}\cdots\int_0^{\infty}(\rho+\beta)^{\frac{1}{2}-s}\\&K_{s-\frac{1}{2}}(2\pi|m|(\rho+\beta))\prod_{j=1}^{n}e^{-2\pi|m|\beta_j}G(2\pi|m|\beta_j)d\beta_1\cdots d\beta_n.\end{aligned}$$

Denote the above integral as $A_m(\rho)$, set $\rho_j=\rho+\beta_{j+1}+\cdots+\beta_n$ for $1\leq j\leq n$, then

$$\begin{aligned}A_m(\rho)=&\int_0^{\infty}\cdots\int_0^{\infty}e^{-2\pi|m|(\beta_2+\cdots+\beta_n)}G(2\pi|m|\beta_2)\cdots G(2\pi|m|\beta_n)\\&\int_0^{\infty}(\rho_1+\beta_1)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|m|(\rho_1+\beta_1))e^{-2\pi|m|\beta_1}G(2\pi|m|\beta_1)d\beta_1\cdots d\beta_n.\end{aligned}$$

We recall that the Weyl fractional integral is defined as

$$h(y;\mu)=\frac{1}{\Gamma(\mu)}\int_y^{\infty}f(x)(x-y)^{\mu-1}dx.\tag{4.5}$$

When $f(x)=x^{-\nu}e^{-\alpha x}K_\nu(\alpha x)$ and $\text{Re}(\mu)>0$,

$$h(y;\mu)=\sqrt{\pi}(2\alpha)^{-\frac{1}{2}\mu-\frac{1}{2}}y^{\frac{1}{2}\mu-\nu-\frac{1}{2}}e^{-\alpha y}W_{-\frac{1}{2}\mu,\nu-\frac{1}{2}\mu}(2\alpha y),\tag{4.6}$$

for $\text{Re}(\alpha y)>0$ (see [3], p. 208, (53)). Here the Whittaker functions (see [4], Vol. I, p. 264) $W_{\kappa,\mu}(x)=e^{-\frac{x}{2}}x^{\frac{c}{2}}\Psi(a,c;x)$, where $a=\frac{1}{2}-\kappa+\mu$, $c=2\mu+1$ and $\Psi(a,c;x)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-xt}t^{a-1}(1+t)^{c-a-1}dt$, $\text{Re}(a)>0$. Thus

$$W_{\kappa,\mu}(x)=e^{-\frac{x}{2}}x^{\mu+\frac{1}{2}}\frac{1}{\Gamma(\frac{1}{2}-\kappa+\mu)}\int_0^{\infty}e^{-xt}t^{-\frac{1}{2}-\kappa+\mu}(1+t)^{-\frac{1}{2}+\kappa+\mu}dt.$$

It is known that (see [4], Vol. I, p. 265, (13))

$$\begin{aligned}K_\nu(x)&=\sqrt{\pi}e^{-x}(2x)^{\nu}\Psi(\frac{1}{2}+\nu,1+2\nu;2x)\\&=\frac{\sqrt{\pi}e^{-x}(2x)^{\nu}}{\Gamma(\nu+\frac{1}{2})}\int_0^{\infty}e^{-2xt}t^{\nu-\frac{1}{2}}(1+t)^{\nu-\frac{1}{2}}dt.\end{aligned}$$

By the above formulas, we have

$$
\begin{aligned}
&A_{1,m}(\rho_1)\\
:=&\int_0^\infty (\rho_1+\beta_1)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|m|(\rho_1+\beta_1))e^{-2\pi|m|\beta_1}G(2\pi|m|\beta_1)d\beta_1\\
=&e^{2\pi|m|\rho_1}\sum_{k=0}^{\infty}\frac{1}{2^k k!}C_{2k}^k(2\pi|m|)^k\int_{\rho_1}^\infty \beta_1^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|m|\beta_1)e^{-2\pi|m|\beta_1}(\beta_1-\rho_1)^k d\beta_1\\
=&\frac{\sqrt{\pi}}{4\pi|m|}\rho_1^{\frac{1}{2}-s}\sum_{k=0}^{\infty}\frac{1}{2^k}C_{2k}^k(\pi|m|\rho_1)^{\frac{k}{2}}W_{-\frac{k}{2}-\frac{1}{2},s-1-\frac{k}{2}}(4\pi|m|\rho_1)\\
=&\frac{\sqrt{\pi}}{4\pi|m|}\frac{1}{\Gamma(s)}(4\pi|m|)^{s-\frac{1}{2}}e^{-2\pi|m|\rho_1}\int_0^\infty e^{-4\pi|m|\rho_1 t}(1+t)^{s-2}t^{s-1}\\
&\times\sum_{k=0}^{\infty}\frac{1}{2^{2k}}C_{2k}^k(1+t)^{-k}dt\\
=&\frac{\sqrt{\pi}}{4\pi|m|\Gamma(s)}(4\pi|m|)^{s-\frac{1}{2}}e^{-2\pi|m|\rho_1}\int_0^\infty e^{-4\pi|m|\rho_1 t}(1+t)^{s-\frac{3}{2}}t^{s-\frac{3}{2}}dt\\
=&\frac{\Gamma(s-\frac{1}{2})}{\sqrt{4\pi|m|}\Gamma(s)}\rho_1^{1-s}K_{s-1}(2\pi|m|\rho_1).
\end{aligned}
$$

In general, we have

$$
\begin{aligned}
A_{j,m}(\rho_j):=&\int_0^\infty(\rho_j+\beta_j)^{\frac{j}{2}-s}K_{s-\frac{j}{2}}(2\pi|m|(\rho_j+\beta_j))e^{-2\pi|m|\beta_j}G(2\pi|m|\beta_j)d\beta_j\\
=&\frac{\Gamma(s-\frac{j}{2})}{\sqrt{4\pi|m|}\Gamma(s-\frac{j}{2}+\frac{1}{2})}\rho_j^{\frac{j+1}{2}-s}K_{s-\frac{j+1}{2}}(2\pi|m|\rho_j),
\end{aligned}
$$

for $1\le j\le n$ and $\rho_n=\rho$. Therefore,

$$
\begin{aligned}
A_m(\rho)=&\prod_{j=1}^{n}\frac{\Gamma(s-\frac{j}{2})}{\sqrt{4\pi|m|}\Gamma(s-\frac{j}{2}+\frac{1}{2})}\rho^{\frac{n+1}{2}-s}K_{s-\frac{n+1}{2}}(2\pi|m|\rho)\\
=&(4\pi|m|)^{-\frac{n}{2}}\frac{\Gamma(s-\frac{n}{2})}{\Gamma(s)}\rho^{\frac{n+1}{2}-s}K_{s-\frac{n+1}{2}}(2\pi|m|\rho).
\end{aligned}
$$

Similarly, the other part of the theorem can be proved. $\square$

5. The Poincaré series for $U(n+1,1)$

The concept of a point-pair invariant was introduced by Selberg [9] who made fascinating use of it. Now, we introduce the following concept.

Definition 5.1. A map $f:\mathbb{H}_{\mathbf{C}}^{n+1}\times\mathbb{H}_{\mathbf{C}}^{n+1}\to\mathbf{C}$ is called *a point-pair invariant associated to a discrete subgroup* $\Gamma\le G(\mathbf{Z})$ if $f(\gamma(P),\gamma(Q))=f(P,Q)$ for all $P,Q\in\mathbb{H}_{\mathbf{C}}^{n+1}$ and $\gamma\in\Gamma$.

THEOREM 5.2. *For* $L=\rho[\beta\frac{\partial^2}{\partial\beta^2}+\rho\frac{\partial^2}{\partial\rho^2}+(\rho+\beta)\frac{\partial^2}{\partial t^2}+n\frac{\partial}{\partial\beta}-n\frac{\partial}{\partial\rho}]$, *set* $f=g(u,v)$, *where*

$$u=u(Z,Z'):=\frac{(t-t')^2+(\rho+\beta-\rho'-\beta')^2}{4\rho\rho'},\quad v=v(Z,Z'):=\frac{\beta\beta'}{\rho\rho'} \tag{5.1}$$

are two point-pair invariants associated to $G(\mathbf{Z})$. *Then*

$$\begin{aligned} Lf=&[(u^2+(\lambda+1)u)\frac{\partial^2}{\partial u^2}+2uv\frac{\partial^2}{\partial u\partial v}+v(v+\lambda)\frac{\partial^2}{\partial v^2}\\ &+((n+2)u+\lambda+1)\frac{\partial}{\partial u}+((n+2)v+n\lambda)\frac{\partial}{\partial v}]g \end{aligned} \tag{5.2}$$

with $\lambda=\frac{\beta'}{\rho'}$, *where* $\gamma(\lambda)=\lambda$ *for* $\gamma\in G(\mathbf{Z})$.

Proof. It is obtained by a straightforward calculation. □

Let $M(u,v,\frac{\partial}{\partial u},\frac{\partial}{\partial v})=[u^2+(\lambda+1)u]\frac{\partial^2}{\partial u^2}+2uv\frac{\partial^2}{\partial u\partial v}+v(v+\lambda)\frac{\partial^2}{\partial v^2}+[(n+2)u+\lambda+1]\frac{\partial}{\partial u}+[(n+2)v+n\lambda]\frac{\partial}{\partial v}+s(n+1-s)$. By transform $x=\frac{u}{\lambda+1},y=\frac{v}{\lambda}$, we have

$$\begin{aligned} M(x,y,\frac{\partial}{\partial x},\frac{\partial}{\partial y})=&x(x+1)\frac{\partial^2}{\partial x^2}+2xy\frac{\partial^2}{\partial x\partial y}+y(y+1)\frac{\partial^2}{\partial y^2}\\ &+[(n+2)x+1]\frac{\partial}{\partial x}+[(n+2)y+n]\frac{\partial}{\partial y}+s(n+1-s). \end{aligned} \tag{5.3}$$

THEOREM 5.3. *Some solutions of the equation* $M(x,y,\frac{\partial}{\partial x},\frac{\partial}{\partial y})g(x,y)=0$ *are as follows:*

$$\begin{aligned} &g_s(x,y)=x^{-a}y^{-b}F_3(a,b;a,b-n+1;2s-n;-x^{-1},-y^{-1}),\\ &g_1(x)=x^{-s}{}_2F_1(s,s;2s-n;-x^{-1}),\\ &g_2(y)=y^{-s}{}_2F_1(s,s-n+1;2s-n;-y^{-1}),\\ &g_3(x+y)=w^{-s}{}_2F_1(s,s-n;2s-n;-(x+y)^{-1}). \end{aligned} \tag{5.4}$$

Proof. At first, we consider the degenerate case.
(1) Set $g(x,y)=g_1(x)$, then one has

$$M(x,\frac{d}{dx})g_1(x)=\{x(x+1)\frac{d^2}{dx^2}+[(n+2)x+1]\frac{d}{dx}+s(n+1-s)\}g_1(x)=0.$$

A solution is $g_1(x)=x^{-s}{}_2F_1(s,s;2s-n;-x^{-1})$.
(2) Set $g(x,y)=g_2(y)$, then one has

$$M(y,\frac{d}{dy})g_2(y)=\{y(y+1)\frac{d^2}{dy^2}+[(n+2)y+n]\frac{d}{dy}+s(n+1-s)\}g_2(y)=0.$$

A solution is $g_2(y)=y^{-s}{}_2F_1(s,s-n+1;2s-n;-y^{-1})$.
(3) Set $g(x,y)=g_3(w)$ with $w=x+y$, then one has

$$M(w,\frac{d}{dw})g_3(w)=\{w(w+1)\frac{d^2}{dw^2}+[(n+2)w+(n+1)]\frac{d}{dw}+s(n+1-s)\}g_3(w)=0.$$

A solution is $g_3(w) = w^{-s} {}_2F_1(s, s-n; 2s-n; -w^{-1})$.

Secondly, we consider the general case. Set $g(x,y) = x^{-a}y^{-b}f(-x^{-1}, -y^{-1})$, then $f(x,y)$ satisfies the following equation:

$$
\begin{aligned}
&x\{x(1-x)\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} + [2(a+b) - n - (2a+1)x]\frac{\partial f}{\partial x} - a^2 f\} + \\
&y\{y(1-y)\frac{\partial^2 f}{\partial y^2} + x\frac{\partial^2 f}{\partial x \partial y} + [2(a+b) - n - (2b-n+2)y]\frac{\partial f}{\partial y} - b(b-n+1)f\} + \\
&(a+b-s)[a+b-(n+1-s)]f = 0.
\end{aligned}
$$

Let

$$
\begin{cases}
x(1-x)\dfrac{\partial^2 f}{\partial x^2} + y\dfrac{\partial^2 f}{\partial x \partial y} + [2(a+b) - n - (2a+1)x]\dfrac{\partial f}{\partial x} - a^2 f = 0, \\
y(1-y)\dfrac{\partial^2 f}{\partial y^2} + x\dfrac{\partial^2 f}{\partial x \partial y} + [2(a+b) - n - (2b-n+2)y]\dfrac{\partial f}{\partial y} - b(b-n+1)f = 0, \\
(a+b-s)[a+b-(n+1-s)] = 0.
\end{cases}
$$

Without loss of generality, we can assume that $a+b=s$. Let $z = f$, $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$, we have

$$
\begin{cases}
x(1-x)r + ys + [2(a+b) - n - (2a+1)x]p - a^2 z = 0, \\
y(1-y)t + xs + [2(a+b) - n - (2b-n+2)y]q - b(b-n+1)z = 0.
\end{cases}
$$

It is known that a solution of the equations

$$
\begin{cases}
x(1-x)r + ys + [\gamma - (\alpha+\beta+1)x]p - \alpha\beta z = 0, \\
y(1-y)t + xs + [\gamma - (\alpha'+\beta'+1)y]q - \alpha'\beta' z = 0.
\end{cases}
$$

is $z = F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y)$(see [2]). Here

$$
\begin{cases}
\gamma = 2(a+b) - n, \\
\alpha + \beta + 1 = 2a + 1, \quad \alpha\beta = a^2, \\
\alpha' + \beta' + 1 = 2b - n + 2, \quad \alpha'\beta' = b(b-n+1).
\end{cases}
$$

i.e., $\alpha = \beta = a$, $\alpha' = b$, $\beta' = b - n + 1$, $\gamma = 2s - n$. Therefore, a family of solutions are

$$g_s(x,y) = x^{-a}y^{-b}F_3(a, b; a, b-n+1; 2s-n; -x^{-1}, -y^{-1}).$$

This completes the proof of Theorem 5.3. □

Definition 5.4. A function $f : \mathbb{H}_{\mathbf{C}}^{n+1} \to \mathbf{C}$ is called *a nonholomorphic automorphic form attached to the unitary group* $U(n+1,1)$ if it satisfies the following three conditions:

(1) f is an eigenfunction of the Laplace-Beltrami operator of $U(n+1,1)$ on $\mathbb{H}_{\mathbf{C}}^{n+1}$;
(2) f is invariant under the modualr group; i.e., $f(\gamma(Z)) = f(Z)$ for all $\gamma \in G(\mathbf{Z})$ and all $Z \in \mathbb{H}_{\mathbf{C}}^{n+1}$.
(3) f has at most polynomial growth at infinity; i.e., there are constants $C > 0$ and k such that $|f(Z)| \le C\rho^k$, as $\rho \to \infty$ uniformly in t, for fixed β.

We denote by $\mathcal{N}(G(\mathbf{Z}),\lambda)$ the space of such nonholomorphic automorphic forms attached to $U(n+1,1)$.

By Theorem 3.4, we have $E(Z;s,0), E(Z;s,1-n)\in\mathcal{N}(G(\mathbf{Z}),s(s-n-1))$ when $\mathrm{Re}(s)>n+1$.

Now, we study the structure of $\mathcal{N}(G(\mathbf{Z}),s(s-n-1))$. The Poincaré series is defined as

$$r(Z,Z';s):=\sum_{\gamma\in G(\mathbf{Z})} g_s(x(Z,\gamma(Z')),y(Z)). \tag{5.5}$$

Theorem 5.5. *$r(Z,Z';s)\in\mathcal{N}(G(\mathbf{Z}),s(s-n-1))$ for Re$(s)>n$, Re$(a)>1$ and Re$(b)>n-1$, where $a+b=s$.*

Proof. Without loss of generality, we can only consider the nondegenerate case.

According to [2], the two variable hypergeometric function F_3 has the following integral representation:

$$\begin{aligned}F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y)=&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\iint_{u\ge0,v\ge0,u+v\le1}u^{\beta-1}\\&v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}dudv,\end{aligned}$$

for $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\beta')>0$ and $\mathrm{Re}(\gamma-\beta-\beta')>0$.

Now, one has

$$\begin{aligned}g_s(x,y)=&\frac{\Gamma(2s-n)}{\Gamma(a)\Gamma(b-n+1)\Gamma(s-1)}\iint_{u\ge0,v\ge0,u+v\le1}u^{a-1}\\&v^{b-n}(1-u-v)^{s-2}(x+u)^{-a}(y+v)^{-b}dudv,\end{aligned}\tag{5.6}$$

for $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>n-1$ and $\mathrm{Re}(s)>1$. On the other hand, $x=x(Z,Z')=(1+\frac{\beta}{\rho})\sigma(z_{n+1},z'_{n+1})$, where $\sigma(z_{n+1},z'_{n+1})=\frac{|z_{n+1}-z'_{n+1}|^2}{4\mathrm{Im}z_{n+1}\mathrm{Im}z'_{n+1}}$, and $y=y(Z)=\frac{\beta}{\rho}$. Therefore, we have

$$\begin{aligned}r(Z,Z';s)=&\frac{\Gamma(2s-n)}{\Gamma(a)\Gamma(b-n+1)\Gamma(s-1)}\iint_{u\ge0,v\ge0,u+v\le1}u^{a-1}v^{b-n}(y+v)^{-b}\\&\times(1-u-v)^{s-2}\sum_{\gamma\in G(\mathbf{Z})}\frac{1}{[(1+\frac{\beta}{\rho})\sigma(z_{n+1},\gamma(z'_{n+1}))+u]^a}dudv.\end{aligned}\tag{5.7}$$

The sum

$$\sum_{\gamma\in G(\mathbf{Z})}\frac{1}{[(1+\frac{\beta}{\rho})\sigma(z_{n+1},\gamma(z'_{n+1}))+u]^{\mathrm{Re}(a)}}\le\sum_{\gamma\in G(\mathbf{Z})}\frac{1}{[\sigma(z_{n+1},\gamma(z'_{n+1}))+u]^{\mathrm{Re}(a)}}.$$

By [8], p. 285, Lemma 1, if $\mathrm{Re}(a)>1$, then the series

$$\sum_{\gamma\in G(\mathbf{Z})}\frac{1}{[1+\sigma(z_{n+1},\gamma(z'_{n+1}))]^{\mathrm{Re}(a)}}$$

is convergent uniformly for z_{n+1}, z'_{n+1} in compact domains. By the same method in [8], one has that if $z_{n+1}\notin G(\mathbf{Z})z'_{n+1}$, then the series in (5.7) is convergent absolutely for $\mathrm{Re}(a)>1$, i.e., $r(Z,Z';s)$ is well-defined for $\mathrm{Re}(a)>1$.

For $g \in G(\mathbf{Z})$, by $x(\gamma(Z), \gamma(Z')) = x(Z, Z')$ and $y(\gamma(Z)) = y(Z)$, we have

$$\begin{aligned} r(g(Z), Z'; s) &= \sum_{\gamma \in G(\mathbf{Z})} g_s(x(g(Z), \gamma(Z')), y(g(Z))) \\ &= \sum_{\gamma \in G(\mathbf{Z})} g_s(x(g(Z), g \circ \gamma(Z')), y(Z)) = \sum_{\gamma \in G(\mathbf{Z})} g_s(x(Z, \gamma(Z')), y(Z)) \\ &= r(Z, Z'; s). \end{aligned}$$

For $\gamma \in G(\mathbf{Z})$, $M(x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})(Z, \gamma(Z')) g_s(x(Z, \gamma(Z')), y(Z)) = 0$. Hence,

$$(L - s(s - n - 1)) g_s(x(Z, \gamma(Z')), y(Z)) = 0.$$

Thus, $(L - s(s - n - 1)) r(Z, Z'; s) = 0$. □

Theorem 5.5 implies the following theorem:

Theorem 5.6. *There exist infinitely many elements in* $\mathcal{N}(G(\mathbf{Z}), s(s - n - 1))$.

6. The Poisson kernel and Eisenstein series for $U(n + 1, 1)$

In this section, we will give the Poisson kernel of L on $\mathbb{H}^{n+1}_{\mathbf{C}}$ and the corresponding Eisenstein series.

Let us give the Iwasawa decomposition of G. $G = KAN = \overline{N}AK$, where

$$A = TA_U T^{-1} = \left\{ a = \begin{pmatrix} I_n & & \\ & e^{\zeta} & \\ & & e^{-\zeta} \end{pmatrix} : \zeta \in \mathbf{R} \right\},$$

$$N = TN_U T^{-1} = \left\{ n = \begin{pmatrix} I_n & 0 & z^t \\ 2i\overline{z} & 1 & t + i|z|^2 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbf{R}, z \in \mathbf{C}^n \right\},$$

$$\overline{N} = T\overline{N_U} T^{-1} = \left\{ \overline{n} = \begin{pmatrix} I_n & iz^t & 0 \\ 0 & 1 & 0 \\ -2\overline{z} & -t - i|z|^2 & 1 \end{pmatrix} : t \in \mathbf{R}, z \in \mathbf{C}^n \right\},$$

and $K = TK_U T^{-1}$.

For $\overline{n} = \begin{pmatrix} I_n & iz^t & 0 \\ 0 & 1 & 0 \\ -2\overline{z} & -t - i|z|^2 & 1 \end{pmatrix}$, $\overline{n}' = \begin{pmatrix} I_n & iw^t & 0 \\ 0 & 1 & 0 \\ -2\overline{w} & -\zeta - i|w|^2 & 1 \end{pmatrix}$ and $a = \begin{pmatrix} I_n & & \\ & \rho^{-\frac{1}{2}} & \\ & & \rho^{\frac{1}{2}} \end{pmatrix}$,

$$[(\overline{n}a)^{-1}]^* = \begin{pmatrix} I_n & 0 & 2\rho^{-\frac{1}{2}} z^t \\ i\overline{z} & \rho^{\frac{1}{2}} & \rho^{-\frac{1}{2}}(t + i|z|^2) \\ 0 & 0 & \rho^{-\frac{1}{2}} \end{pmatrix}.$$

For $Z \in \mathbb{H}_{\mathbf{C}}^{n+1}$, consider $Z' = (z', z'_{n+1}) = (-\frac{z}{\sqrt{\rho}}, i\frac{y_{n+1}}{\rho})$, $\mathrm{Im} z'_{n+1} - |z'|^2 = 1$, so $Z' \in \mathbb{H}_{\mathbf{C}}^{n+1}$. Thus

$$[(\overline{n}a)^{-1}]^*(-\frac{z}{\sqrt{\rho}}, i\frac{y_{n+1}}{\rho}) = (z, t + iy_{n+1}) = Z. \tag{6.1}$$

$$a^{-1}\overline{n}^{-1}\overline{n}' = \begin{pmatrix} I_n & -i(z^t - w^t) & 0 \\ 0 & \rho^{\frac{1}{2}} & 0 \\ 2\rho^{-\frac{1}{2}}(\overline{z} - \overline{w}) & \rho^{-\frac{1}{2}}[t - \zeta - i(|z|^2 + |w|^2 - 2\overline{z}w^t)] & \rho^{-\frac{1}{2}} \end{pmatrix}. \tag{6.2}$$

For $a \in A$ and $n \in N$ and $k = Tk_U T^{-1} = \frac{1}{2i}\begin{pmatrix} A_1 & * \\ A_2 & * \end{pmatrix}$, where $k_U = \begin{pmatrix} A & \\ & D \end{pmatrix}$, $A = (a_{ij}) \in U(n+1)$, $D \in U(1)$, A_1 is an $(n+1)\times(n+1)$-matrix and A_2 is an $1\times(n+1)$-matrix. $an = \begin{pmatrix} B & * \\ 0 & * \end{pmatrix}$ with $B = \begin{pmatrix} I_n & 0 \\ 2ie^{\zeta}\overline{z} & e^{\zeta} \end{pmatrix}$. $kan = \frac{1}{2i}\begin{pmatrix} A_1 B & * \\ A_2 B & * \end{pmatrix}$, where $A_1 B = \begin{pmatrix} * & * \\ * & ie^{\zeta}(a_{n+1,n+1} + D) \end{pmatrix}$ and $A_2 B = (*, (-a_{n+1,n+1} + D)e^{\zeta})$.

For $g = (g_{ij}) \in G$, we have

$$g_{n+1,n+1} = \frac{1}{2}e^{\zeta}(a_{n+1,n+1} + D), \quad g_{n+2,n+1} = \frac{1}{2i}e^{\zeta}(-a_{n+1,n+1} + D).$$

$D \in U(1)$ implies that $e^{2\zeta} = |g_{n+1,n+1} + ig_{n+2,n+1}|^2$. Therefore, the Poisson kernel

$$\begin{aligned} P(Z, W) &= |\rho^{\frac{1}{2}} + i\rho^{-\frac{1}{2}}[t - \zeta - i(|z|^2 + |w|^2 - 2\overline{z}w^t)]|^{-2} \\ &= \frac{\rho}{|\rho + |z - w|^2 + i(t - \zeta - 2\mathrm{Im}\overline{z}w^t)|^2}, \end{aligned} \tag{6.3}$$

where $W = (w_1, \cdots, w_n, \zeta, 0) \in \mathbf{C}^n \times \mathbf{R}$. We have $LP(Z, W)^s = s(s - n - 1)P(Z, W)^s$. In fact, by Helgason's conjecture, which was proved by Kashiwara et al. in [6], that the eigenfunctions on Riemannian symmetric spaces can be represented as Poisson integrals of their hyperfunction boundary values.

In the theory of automorphic forms, the rigid property is essential, it is determined by the discrete subgroup. $G(\mathbf{Z})$ acts on $\mathbb{H}_{\mathbf{C}}^{n+1}$, not $\mathbf{C}^n \times \mathbf{R} \times (0, \infty)$, although they are diffeomorphic. The boundary of $\mathbb{H}_{\mathbf{C}}^{n+1}$ is

$$\partial \mathbb{H}_{\mathbf{C}}^{n+1} = \{(z, y_{n+1}) \in \mathbf{C}^n \times \mathbf{R} : y_{n+1} - |z|^2 = 0\} \times \{t : t \in \mathbf{R}\}.$$

The first one is called the constraint boundary, the second is called the free boundary. What we need is the second one. Set $\mathcal{H} := \{t + i\rho : t \in \mathbf{R}, \rho > 0\}$ and let $\Omega(\Gamma, \mathcal{H})$ be the region of discontinuity of $\Gamma \leq G(\mathbf{Z})$ corresponding to $\mathcal{H}$. For $Z = (z, t, \rho) = (z, z_{n+1}) \in \mathbb{H}_{\mathbf{C}}^{n+1}$ and $W = (w, \zeta, 0) = (w, w_{n+1}) \in \partial\mathbb{H}_{\mathbf{C}}^{n+1}$, where $\rho(W) = \mathrm{Im} w_{n+1} - |w|^2 = 0$. If $w = 0$, $\rho(W) = 0$ implies that $\mathrm{Im} w_{n+1} = |w|^2 = 0$, i.e., $w_{n+1} = \zeta \in \mathbf{R}$. The Poisson kernel associated with the free boundary is:

$$P(Z, \zeta) := \frac{\rho}{(\rho + \beta)^2 + (t - \zeta)^2}. \tag{6.4}$$

Theorem 6.1. *$LP(Z, \zeta)^s = s(s - n - 1)P(Z, \zeta)^s$.*

THEOREM 6.2. *$P(\gamma(Z),\gamma(\zeta))|\gamma'(\zeta)| = P(Z,\zeta)$, for $\gamma \in G(\mathbf{Z})$.*

We define the Eisenstein series

$$E(Z,\zeta;s) := \sum_{\gamma\in\Gamma} P(\gamma(Z),\zeta)^s, \quad \mathrm{Re}(s) > \delta(\Gamma), \tag{6.5}$$

where $\delta(\Gamma)$ is the critical exponent of Γ.

THEOREM 6.3. *The Eisenstein series satisfies the following properties:*

$$E(\gamma(Z),\zeta;s) = E(Z,\zeta;s), \quad E(Z,\gamma(\zeta);s) = |\gamma'(\zeta)|^{-s}E(Z,\zeta;s),$$
$$LE(Z,\zeta;s) = s(s-n-1)E(Z,\zeta;s). \tag{6.6}$$

The scattering matrix is defined as

$$S(\zeta,\eta;s) := \sum_{\gamma\in\Gamma} \frac{|\gamma'(\zeta)|^s}{|\gamma(\zeta)-\eta|^{2s}}, \quad \mathrm{Re}(s) > \delta(\Gamma), \tag{6.7}$$

where $\zeta,\eta \in \Omega(\Gamma,\mathcal{H})$. It describes the normalized free boundary behaviour of the Eisenstein series

$$S(\zeta,t;s) = \lim_{\rho\to 0}\lim_{\beta\to 0} \rho^{-s}E(Z,\zeta;s).$$

For $\mathrm{Re}(s) > n+1$, we define

$$G_0(Z,Z';s) := r_s(u(Z,Z')), \quad \text{and} \quad G(Z,Z';s) := \sum_{\gamma\in\Gamma} G_0(\gamma(Z),Z';s),$$

where $r_s(u) = g_1(x)$ in Theorem 5.3. Then, we have

$$\lim_{\rho'\to 0}\lim_{\beta'\to 0}(\rho')^{-s}G_0(Z,Z';s) = c(s)P(Z,t')^s.$$

Consequently,

$$\lim_{\rho'\to 0}\lim_{\beta'\to 0}(\rho')^{-s}G(Z,Z';s) = c(s)E(Z,t';s).$$

7. The modular forms and modular varieties on $U(n+1,1)$

In his paper [10], Wirthmüller gave the Jacobi modular forms associated to the root systems. Now we give the definition of modular forms on $U(n+1,1)$ associated to $G(\mathbf{Z})$.

Definition 7.1. A modular form on $U(n+1,1)$ associated with $G(\mathbf{Z})$ is a function $\phi : \mathbb{H}_{\mathbf{C}}^{n+1} \to \mathbf{C}$ satisfying the following transform equations:

(1) $\phi(\frac{z}{cz_{n+1}+d}, \frac{az_{n+1}+b}{cz_{n+1}+d}) = (cz_{n+1}+d)^k e^{2\pi i m c(z_1^2+\cdots+z_n^2)/(cz_{n+1}+d)}\phi(z,z_{n+1})$.
(2) $\phi(wz,z_{n+1}) = \phi(z,z_{n+1})$ for all $w \in S_n$, where S_n is the symmetric group of n-order.
(3) $\phi(z,z_{n+1})$ is a locally bounded function as $\mathrm{Im} z_{n+1} \to \infty$.

According to [10], for $Z=(z,z_{n+1})\in\mathbb{H}_{\mathbf{C}}^{n+1}$ and $w\in\mathbf{C}$, we have

$$\gamma(z,z_{n+1},w)=(\frac{z}{cz_{n+1}+d},\frac{az_{n+1}+b}{cz_{n+1}+d},w-c\frac{z_1^2+\cdots+z_n^2}{cz_{n+1}+d}).$$

The modular forms on $U(n+1,1)$ can be written as follows

$$\phi\circ\gamma(z,z_{n+1})=(\frac{d\gamma(z_{n+1})}{dz_{n+1}})^{-\frac{k}{2}}e^{-2\pi im(\gamma(w)-w)}\phi(z,z_{n+1}). \tag{7.1}$$

Set $\mathcal{M}_{n+1}=G(\mathbf{Z})\backslash\mathbb{H}_{\mathbf{C}}^{n+1}$. It is well known that $\mathcal{M}_1\cong\mathbf{C}$, the coarse moduli space of complex elliptic curves.

Now, we consider the modular functions on $U(n+1,1)$.

$$(\phi|_{k,m}\gamma)(z,z_{n+1}):=(cz_{n+1}+d)^{-k}e^m(\frac{-c(z_1^2+\cdots+z_n^2)}{cz_{n+1}+d})\phi\circ\gamma(z,z_{n+1}), \tag{7.2}$$

for $\gamma\in G(\mathbf{Z})$, where $e^m(x):=e^{2\pi imx}$.

The Eisenstein series for $U(n+1,1)$ with weight k and index m is defined as

$$E_{k,m}(z,z_{n+1}):=\sum_{\gamma\in G(\mathbf{Z})_\infty\backslash G(\mathbf{Z})}(1|_{k,m}\gamma)(z,z_{n+1}). \tag{7.3}$$

Explicitly, this is

$$E_{k,m}(z,z_{n+1})=\frac{1}{2}\sum_{c,d\in\mathbf{Z},(c,d)=1}(cz_{n+1}+d)^{-k}e^m(\frac{-c(z_1^2+\cdots+z_n^2)}{cz_{n+1}+d}).$$

It is clear that

$$E_{k,m}(z,z_{n+1}+1)=E_{k,m}(z,z_{n+1}).$$

$$E_{k,m}(\frac{z}{z_{n+1}},-\frac{1}{z_{n+1}})=\frac{1}{2}z_{n+1}^k\sum_{c,d\in\mathbf{Z},(c,d)=1}(dz_{n+1}-c)^{-k}e^m(\frac{-c(z_1^2+\cdots+z_n^2)}{-cz_{n+1}+dz_{n+1}^2}).$$

By the identity:

$$\frac{-c}{-cz_{n+1}+dz_{n+1}^2}-\frac{1}{z_{n+1}}=\frac{-d}{dz_{n+1}-c},$$

we have

$$E_{k,m}(\frac{z}{z_{n+1}},-\frac{1}{z_{n+1}})=z_{n+1}^ke^m(\frac{z_1^2+\cdots+z_n^2}{z_{n+1}})E_{k,m}(z,z_{n+1}).$$

The set of modular forms on $U(n+1,1)$ is denoted as $M_{k,m}(G(\mathbf{Z}))$. It is obvious that $E_{k,m}(z,z_{n+1})\in M_{k,m}(G(\mathbf{Z}))$.

Now, let us define a family of modular functions for $U(n+1,1)$ of weight 0 and index 0, the j-invariants associated to $U(n+1,1)$:

$$j_m(z,z_{n+1}):=\frac{1728g_{2,m}(z,z_{n+1})^3}{\Delta_m(z,z_{n+1})}, \tag{7.4}$$

where $g_{2,m_1}(z,z_{n+1}):=\frac{4}{3}\pi^4E_{4,m_1}(z,z_{n+1})$, $g_{3,m_2}(z,z_{n+1}):=\frac{8}{27}\pi^6E_{6,m_2}(z,z_{n+1})$, and $\Delta_m(z,z_{n+1}):=g_{2,m}(z,z_{n+1})^3-27g_{3,\frac{3}{2}m}(z,z_{n+1})^2$. In fact, for $\gamma\in G(\mathbf{Z})$,

$$j_m(\gamma(z,z_{n+1}))=j_m(z,z_{n+1}). \tag{7.5}$$

Therefore, j_m is a modular function on the modular variety $\mathcal{M}_{n+1}$.

If $z=0$ then $\mathcal{M}_{n+1}$ degenerates to $\mathcal{M}_1$ and $j_m(0,z_{n+1})=j(z_{n+1})$. It is well known that for every $c\in\mathbf{C}$, $j(z_{n+1})=c$ has exactly one solution. Thus, $j(z_{n+1})$ is an analytic isomorphism from $\mathcal{M}_{n+1}$ to $\mathbf{C}$. Therefore, $j_m:\mathcal{M}_{n+1}\to\mathbf{C}$ is a surjective morphism. Now, we complete the proof of Main Theorem.

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